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Metrics of Static Spheroidal Charged Dust Distributions

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Two sets of solutions for a static incoherent charge distribution with spheroidal symmetry are presented. In one set, the charge distribution can be bounded and the charge-to-mass-density ratio is unity everywhere in relativistic units. In the other, the distribution pervades the entire space and the ratio of charge to mass-density also varies at different points.

1. INTRODUCTION

In this communication we present two sets of exact solutions of incoherent static charge distributions having the symmetry of oblate spheroids. The first set represents the distribution in the form of finite oblate spheroids and the charge-to-mass-density ratio is unity everywhere in relativistic units. The exterior electrovac solution is also presented. The solution may be adjusted to represent any spheroidal charged dust distribution of finite size. The value of charge-to-mass-density ratio is quite consistent with the known theorem (De and Raychaudhuri, 1968): If any incoherent charge distribution in equilibrium is bound by an electric equipotential surface then the charge-to-mass-density ratio must be unity everywhere in relativistic units. The validity of the theorem has been verified for finite spherically symmetric incoherent charged matter in equilibrium (Bonnor, 1965). In case of two other simple symmetries this theorem cannot be verified as a static charged dust distribution cannot exist with cylindrical or plane symmetries (Som, 1964, 1967; De, 1973).

The second set also represents an incoherent charge distribution, but the distribution cannot be bounded. It is quite interesting to note that the

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charge-to-mass-density ratio is not unity in this case and it varies in the distribution.

Incidentally, there are a few solutions with the spheroidal symmetry both for exterior electrovac space and for exterior matter-free space with gravitational field only (Misra, 1960; Misra, 1962; Zipoy, 1966; Bannerjee and Das, 1976). Ours is an attempt to find the solution in the source region.

2. THE BOUND DISTRIBUTION

Following Weyl (Synge, 1971), the general static axially symmetric metric for incoherent charged matter can be taken as

$$dS^2 = e^{\nu}\, dt^2 - e^{\alpha}(d\rho^2 + dZ^2) - e^{-\nu}\rho^2\, d\varphi^2 \tag{2.1}$$

where ν and α are functions of ρ and Z only.

However, we consider here "oblate spheroidal coordinates" and so we make a transformation of the ρ and Z coordinates to

$$\rho = a \cosh u \cos\theta \qquad \text{and} \qquad Z = a \sinh u \sin\theta \tag{2.2}$$

where a is a constant. The illustration of oblate spheroidal coordinates may be seen elsewhere (Zipoy, 1966). The u-constant surfaces are found to be the surface of the spheroids.

With the above transformation, the metric (2.1) takes the form

$$\begin{aligned} dS^2 = {} & e^{\nu}\, dt^2 - a^2 e^{\alpha}(\sinh^2 u + \sin^2\theta)(du^2 + d\theta^2) \\ & - a^2 e^{-\nu}\cosh^2 u \cos^2\theta\, d\varphi^2 \end{aligned} \tag{2.3}$$

where $\nu, \alpha = \nu, \alpha(u, \theta)$.

Since the static charged matter has spheroidal symmetry, the existing electric field is along the u directions (denoted by the suffix 1). Then only F_{01} or F^{01} components of the electromagnetic tensor will exist, and let us write

$$E^2 = -(\tfrac{1}{8}\pi)F^{01}F_{01} \tag{2.4}$$

so that E will give the electromagnetic energy density.

Now, considering the Einstein–Maxwell field equations one can find that only the following four independent expressions remain:

$$\nu_1\nu_2 + (\alpha_1 + \nu_1)\tan\theta - (\alpha_2 + \nu_2)\tanh u = 0 \tag{2.5a}$$

$$\nu_2{}^2 + (\alpha_{11} + \nu_{11}) + (\alpha_{22} + \nu_{22}) + (\alpha_2 + \nu_2)\tan\theta + (\alpha_1 + \nu_1)\tanh u = 0 \tag{2.5b}$$

$$\begin{aligned} 8\pi E^2 = {} & \frac{e^{-\alpha}}{4a^2(\sinh^2 u + \sin^2\theta)}[(\alpha_{11} + \nu_{11}) + (\alpha_{22} + \nu_{22}) + \nu_1{}^2 \\ & - (\alpha_1 + \nu_1)\tanh u - (\alpha_2 + \nu_2)\tan\theta] \end{aligned} \tag{2.5c}$$

$$4\pi\rho = \frac{e^{-\alpha}}{4a^2(\sinh^2 u + \sin^2\theta)}[-\nu_1{}^2 - (\alpha_{11} - \nu_{11}) - (\alpha_{22} - \nu_{22}) + (\alpha_1 + 3\nu_1)\tanh u + (\alpha_2 - \nu_2)\tan\theta] \quad (2.5d)$$

where ρ is the matter density.

In order to avoid a singularity at the center of the coordinate system, it is obvious from equation (2.1) that the following condition must be satisfied:

$$\lim_{\rho\to 0}\left[2\pi e^{\alpha/2}\rho = \int_{\varphi=0}^{2\pi} e^{-\nu/2}\rho\, d\varphi\right]$$

This amounts to

$$\lim_{\rho\to 0}(\alpha + \nu) = 0 \quad (2.6a)$$

Hence, if one wants to solve for α and ν from equations (2.5a) and (2.5b) the condition (2.6a) must be satisfied to have a regular solution at the center.

We now make the simplest assumption regarding equation (2.6a), i.e., let us assume

$$\alpha + \nu = 0 \quad (2.6b)$$

everywhere. Under the condition (2.6b), one finds from (2.5a) and (2.5b) that the only consistent solution is $\nu_2 = 0$.

Hence,

$$\nu = \nu(u) \quad (2.7)$$

Then from equations (2.5c) and (2.5d),

$$8\pi E^2 = \frac{e^{\nu}\nu_1{}^2}{4a^2(\sinh^2 u + \sin^2\theta)} \quad (2.8a)$$

and

$$4\pi\rho = \frac{e^{\nu}(2\nu_{11} - \nu_1{}^2 + 2\nu_1\tanh u)}{4a^2(\sinh^2 u + \sin^2\theta)} \quad (2.8b)$$

The internal metric is

$$dS^2 = e^{\nu}\,dt^2 - e^{-\nu}\{a^2[(\sinh^2 u + \sin^2\theta)(du^2 + d\theta^2) + \cosh^2 u\cos^2\theta\, d\varphi^2]\} \quad (2.8c)$$

ν must be chosen in such a way that ρ and E^2 must be regular everywhere and as $u \to 0$, E^2 must vanish.

The external electrovac space is given by the metric

$$dS^2 = [1 + c\cot^{-1}(e^{\nu})]^{-2}\,dt^2 - [1 + c\cot^{-1}(e^{\nu})]^2[a^2(d\rho^2 + dZ^2 + \rho^2\, d\varphi^2)], \quad (2.9)$$

obviously, the external metric is asymptotically flat at $u \to \infty$, i.e., at infinite distance from the source.

Across the boundary, $g_{\mu\nu}$ and $g'_{\mu\nu}$ must be continuous (O'Brien and Synge, 1952); so for any suitable form of ν, the arbitrary constant c and the corresponding boundary of the source surface u may be evaluated. In general the value of the arbitrary constant c and the value of u at the boundary cannot be expressed in a closed form in terms of e^ν and ν_1 at the boundary. But these values can be numerically calculated for a particular suitable form of e^ν of the internal metric. In the following we give an example where c and the value of u at the boundary can be expressed in a closed form.

Let us take

$$e^\nu = \frac{e^{K \tan^{-1}(e^u)}}{[1 + e^{(K/2)\tan^{-1}(e^u)}]^2} \tag{2.10a}$$

where the arbitrary constant K is positive. Then from the continuity of e^ν and $(d/du)(e^\nu)$ across the boundary one can see that at the boundary of the distribution

$$u = \ln [\cot 2/K] \tag{2.10b}$$

and the constant

$$c = \frac{K}{2} e^{-(K\pi/4 - 1)} \tag{2.10c}$$

One can find in a paper by Bonnor and Wickramasuriya (1972) another example where another form of e^ν has been taken.

Lastly, the charge-to-mass-density ratio σ/ρ may be calculated from the equation of motion

$$\frac{d^2x^\mu}{dS^2} + \Gamma^\mu_{\alpha\beta} v^\alpha v^\beta = -\frac{\sigma}{\rho} F^\alpha{}_\beta v^\beta \tag{2.11}$$

It can be found out that $\sigma/\rho = \pm 1$ for the internal space given by equation (2.8c).

Since $F_{\mu\nu} = \Phi_{\mu,\nu} - \Phi_{\nu,\mu}$ and in our case only the F_{01} component exists, $F_{01} = \Phi_{0,1}$, where Φ_0 is the electric potential. Then a little examination of equation (2.8a) shows that u-constant surfaces are also Φ_0-constant surfaces, i.e., electric equipotential surfaces. Hence the above solution satisfies the theorem mentioned in the Introduction (De and Raychaudhuri, 1968).

3. THE COSMOLOGICAL DISTRIBUTION

Let us write $\nu + \alpha = \lambda$. Then equations (2.5a)–(2.5d) may be rewritten as

$$\nu_1\nu_2 + \lambda_1 \tan\theta - \lambda_2 \tanh u = 0 \tag{3.1a}$$

$$\nu_2{}^2 + (\lambda_{11} + \lambda_{22}) + \lambda_1 \tanh u + \lambda_2 \tan\theta = 0 \tag{3.1b}$$

$$8\pi E^2 = \frac{e^{\nu-\lambda}}{4a^2(\sinh^2 u + \sin^2\theta)}[\nu_1{}^2 + (\lambda_{11} + \lambda_{22}) - \lambda_1 \tanh u - \lambda_2 \tan\theta] \tag{3.1c}$$

and,

$$4\pi\rho = \frac{e^{\nu-\lambda}}{4a^2(\sinh^2 u + \sin^2\theta)} \times [2(\nu_{11} + \nu_{22}) - 2(\lambda_{11} + \lambda_{22}) - \nu_1{}^2 - \nu_2{}^2 + 2\nu_1 \tanh u - 2\nu_2 \tan\theta] \tag{3.1d}$$

Let us assume

$$\lambda_1 = 0 \tag{3.2}$$

Further, if we assume that ν is in linear combination of a function of u and a function of θ, then from equation (3.1a), we can write

$$\nu_1 = \pm \tanh u \qquad \text{and} \qquad \nu_2 = \pm \lambda_2$$

Hence, we get two solutions for ν,

(i) $\nu = \ln\cosh u + \lambda$ (3.3a)

(ii) $\nu = -\ln\cosh u - \lambda$ (3.3b)

However, one can see from equation (3.1b) that in either case the solution for λ remains the same, which is

$$e^\lambda = (C\sin\theta + D) \tag{3.4}$$

where C and D are arbitrary constants.

We are now going to find the first set of solutions. With equations (3.4), (3.3a), (3.1c), and (3.1d), we get

$$8\pi E^2 = \frac{\cosh u}{4a^2(\sinh^2 u + \sin^2\theta)}\left[\tanh^2 u - \frac{2C\sin\theta}{(C\sin\theta + D)} + \frac{C^2\cos^2\theta}{(C\sin\theta + D)^2}\right] \tag{3.5a}$$

$$4\pi\rho = \frac{\cosh u}{4a^2(\sinh^2 u + \sin^2\theta)}\left[2 - \tanh^2 u - \frac{2C\sin\theta}{(C\sin\theta + D)} - \frac{C^2\cos^2\theta}{(C\sin\theta + D)^2}\right] \tag{3.5b}$$

and

$$dS^2 = \cosh u(C\sin\theta + D)\,dt^2 - \frac{a^2}{\cosh u}(\sinh^2 u + \sin^2\theta)(du^2 + d\theta^2) - a^2\frac{\cosh u\cos^2\theta\, d\varphi}{(C\sin\theta + D)} \tag{3.5c}$$

The expression (3.5b) shows that the distribution can never be contained within a spheroid. Besides, a little examination of the equation shows that there cannot even exist any pole where the matter density will vanish provided $(C/D)^2 < 1$. Further, to keep the matter density and the electric energy density positive everywhere, C and D must be of opposite sign, besides $|D| > |C|$. Obviously at infinity (i.e., $u \to \infty$), both the matter density and the electric energy density tends to zero. Hence, the space-time given by the metric (3.5c) (with the condition that C and D must be of opposite sign and $|C| < |D|$) has a continuous incoherent charged matter distribution extending up to infinity. Using equation (2.11), we can find that the ratio

$$\left|\frac{\sigma}{\rho}\right| = \tanh u\left[\tanh^2 u - \frac{2C\sin\theta}{(C\sin\theta + D)} + \frac{C^2\cos^2\theta}{(C\sin\theta + D)^2}\right]^{-1/2}$$

is zero only at the center and at best it can be made unity only over a θ-constant surface.

The above is a physical cosmological solution for an incoherent charged matter distribution in equilibrium.

It can be shown that the other case, i.e., $\nu_1 = -\tanh u$ and $\nu_2 = -\lambda_2$, does not represent any physically permissible solution with positive matter and energy density everywhere.

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